

ALL 2-POSITIVE LINEAR MAPS FROM $M_3(\mathbb{C})$ TO $M_3(\mathbb{C})$ ARE DECOMPOSABLE

YU YANG, DENNY H. LEUNG, AND WAI-SHING TANG

Abstract

Following an idea of Choi, we obtain a decomposition theorem for k -positive linear maps from $M_m(\mathbb{C})$ to $M_n(\mathbb{C})$, where $2 \leq k < \min\{m, n\}$. As a consequence, we give an affirmative answer to Kye's conjecture (also solved independently by Choi) that every 2-positive linear map from $M_3(\mathbb{C})$ to $M_3(\mathbb{C})$ is decomposable.

Keywords: positive maps between low-dimensional matrix algebras, k -positivity, decomposability, Schmidt number, PPT bound entangled states.

1. INTRODUCTION

Let $M_n(\mathbb{C})$ be the algebra of all $n \times n$ matrices over the complex field \mathbb{C} . We say that a matrix A in $M_n(\mathbb{C})$ is positive semi-definite, and write $A \geq 0$, if A is hermitian and all eigenvalues of A are non-negative. Denote by $M_n^+(\mathbb{C})$ the set of all positive semi-definite matrices in $M_n(\mathbb{C})$, and by $B(M_m(\mathbb{C}), M_n(\mathbb{C}))$ the space of all linear maps from $M_m(\mathbb{C})$ to $M_n(\mathbb{C})$.

Definition 1.1. A linear map ϕ from $M_m(\mathbb{C})$ to $M_n(\mathbb{C})$ is called positive if $\phi(M_m^+(\mathbb{C})) \subseteq M_n^+(\mathbb{C})$.

The identity map on $M_n(\mathbb{C})$ and the transpose map on $M_n(\mathbb{C})$ are denoted by id_n and τ_n respectively.

Definition 1.2. A map ϕ is called k -positive if the map $id_k \otimes \phi : M_k(M_m(\mathbb{C})) \rightarrow M_k(M_n(\mathbb{C}))$ is positive. Similarly, a map ϕ is called k -copositive if the map $\tau_k \otimes \phi : M_k(M_m(\mathbb{C})) \rightarrow M_k(M_n(\mathbb{C}))$ is positive.

If a map is k -positive (resp. k -copositive) for every k , it is called completely positive (resp. completely copositive). A positive map is called decomposable if it can be written as the sum of a completely positive map and a completely copositive map.

In [3], Cho, Kye and Lee introduced the generalized Choi maps and discussed the conditions for the maps to be k -positive or decomposable. For generalized Choi's map in $B(M_3(\mathbb{C}), M_3(\mathbb{C}))$, they showed that 2-positivity or 2-copositivity implies decomposability. It is natural to ask whether this property holds for every 2-positive or 2-copositive map in $B(M_3(\mathbb{C}), M_3(\mathbb{C}))$ (see [8, page 1330002-11]).

Conjecture 1.1. Every 2-positive (respectively 2-copositive) linear map in $B(M_3(\mathbb{C}), M_3(\mathbb{C}))$ is decomposable.

Let us recall some useful definitions. Denote by \mathcal{H}_A and \mathcal{H}_B two Hilbert spaces with $\dim(\mathcal{H}_A) = m$ and $\dim(\mathcal{H}_B) = n$, respectively.

Definition 1.3. Every vector $z \in \mathcal{H}_A \otimes \mathcal{H}_B$ has a canonical expansion $z = \sum_{i=1}^m e_i \otimes z_i$, where $\{e_i\}_{i=1}^m$ is a basis for \mathcal{H}_A and $z_i \in \mathcal{H}_B$ for $i = 1, 2, \dots, m$. The *Schmidt rank* $SR(z)$ of the vector z is defined to be the dimension of $\text{span}(\{z_1, \dots, z_m\})$.

Definition 1.4. [14, 10] Consider the density matrix ρ for a quantum state in a bipartite system $\mathcal{H}_A \otimes \mathcal{H}_B$. The *Schmidt number* of the density matrix (or the state) ρ is defined by

$$SN(\rho) = \min \left\{ \max_k SR(z_k) \right\},$$

where the minimum is taken over all possible decompositions

$$\rho = \sum_k p_k \cdot z_k z_k^*$$

with z_k being vectors in $\mathcal{H}_A \otimes \mathcal{H}_B$ and $p_k > 0$, $\sum_k p_k = 1$.

Sanpera, Bruß and Lewenstein in [10] formulated the following conjecture and presented strong evidence of its validity for some special cases.

Conjecture 1.2. *All bound entangled states with positive partial transpose in $\mathbb{C}^3 \otimes \mathbb{C}^3$ have Schmidt number 2.*

There is a diagram of dual cone relations between quantum states and positive maps (see [11, 7, 6, 8]). Let us consider the duality between the space $M_m(\mathbb{C}) \otimes M_n(\mathbb{C})$ and the space $B(M_m(\mathbb{C}), M_n(\mathbb{C}))$. Let E_{ij} be the canonical matrix units in $M_m(\mathbb{C})$. For $A = \sum_{i,j=1}^m E_{ij} \otimes A_{ij} \in M_m(\mathbb{C}) \otimes M_n(\mathbb{C})$ and a linear map $\phi \in B(M_m(\mathbb{C}), M_n(\mathbb{C}))$, define a bilinear form:

$$\langle A, \phi \rangle = \sum_{i,j=1}^m \text{Tr}(\phi(E_{ij})A_{ij}^t).$$

Note that for two normed real spaces X and Y which are dual to each other with respect to a bilinear form $\langle \cdot, \cdot \rangle$, the dual cone for a subset C of X is defined as $C^\circ = \{y \in Y : \langle x, y \rangle \geq 0 \text{ for each } x \in C\}$. Denote by $\mathbb{P}_k[m, n]$ and $\mathbb{P}^k[m, n]$ the set of all k -positive maps and the set of all k -copositive maps in $B(M_m(\mathbb{C}), M_n(\mathbb{C}))$, respectively. Define convex cones $\mathbb{V}_k[m, n]$ and $\mathbb{V}^k[m, n]$ in $M_m(\mathbb{C}) \otimes M_n(\mathbb{C})$ as

$$\begin{aligned} \mathbb{V}_k[m, n] &= \{zz^* : SR(z) \leq k, z \in \mathbb{C}^m \otimes \mathbb{C}^n\}^{\circ\circ}, \\ \mathbb{V}^k[m, n] &= \{(zz^*)^\Gamma : SR(z) \leq k, z \in \mathbb{C}^m \otimes \mathbb{C}^n\}^{\circ\circ}. \end{aligned}$$

Here Γ is an operation called partial transposition that acts as transposition only on the first part of a tensor product. By the dual correspondence between maps and states, we have the following diagram:

$$\begin{array}{ccccccc} \mathbb{V}_1 & \subsetneq & \cdots & \mathbb{V}_k & \subsetneq & \mathbb{V}_{m \wedge n} & = (M_m(\mathbb{C}) \otimes M_n(\mathbb{C}))^+ \\ \updownarrow & & & \updownarrow & & \updownarrow & \\ \mathbb{P}_1 & \supsetneq & \cdots & \mathbb{P}_k & \supsetneq & \mathbb{P}_{m \wedge n} & \cong (M_m(\mathbb{C}) \otimes M_n(\mathbb{C}))^+ \end{array},$$

where $m \wedge n = \min\{m, n\}$. A similar diagram holds in case of copositivity:

$$\begin{array}{ccccccc} \mathbb{V}^1 & \subsetneq & \cdots & \mathbb{V}^k & \subsetneq & \mathbb{V}^{m \wedge n} & \\ \updownarrow & & & \updownarrow & & \updownarrow & \\ \mathbb{P}^1 & \supsetneq & \cdots & \mathbb{P}^k & \supsetneq & \mathbb{P}^{m \wedge n} & \cong (M_m(\mathbb{C}) \otimes M_n(\mathbb{C}))^+ \end{array}.$$

Denote by $\mathbb{D}[m, n]$ the convex cone given by $\mathbb{P}_{m \wedge n} + \mathbb{P}^{m \wedge n}$. Correspondingly, denote by $\mathbb{T}[m, n]$ the cone of states given by $\mathbb{V}_{m \wedge n} \cap \mathbb{V}^{m \wedge n}$.

One can check that $(\mathbb{D}[m, n], \mathbb{T}[m, n])$ is a dual pair defined through the bilinear pairing between $B(M_m(\mathbb{C}), M_n(\mathbb{C}))$ and $M_m(\mathbb{C}) \otimes M_n(\mathbb{C})$. It is natural to ask where should we locate the pair $(\mathbb{D}[m, n], \mathbb{T}[m, n])$ in the above two diagrams. Moreover, it follows from duality in the diagrams that Conjecture 1.1 and Conjecture 1.2 are equivalent.

This paper is organized as follows. In Section 2, we will give a decomposition theorem in order to relate a k -positive map in $B(M_m(\mathbb{C}), M_n(\mathbb{C}))$ to a $(k-1)$ -positive map which actually resides in $B(M_{m-1}(\mathbb{C}), M_n(\mathbb{C}))$. In Section 3, we will give an affirmative answer to Conjecture 1.1 and hence Conjecture 1.2 too (see Theorem 3.2 and Corollary 3.6). Examples are provided throughout to illustrate certain aspects of the results obtained in these sections.

2. A DECOMPOSITION FOR ALL k -POSITIVE / k -COPOSITIVE MAPS

Our approach towards Conjecture 1.1 is to peel off a completely positive map from a 2-positive map. That is, find a completely positive map which is dominated by the 2-positive map. Moreover, the dimension of the space where the remaining map resides is reduced. Indeed, this is a dimension-lowering trick. For a positive linear map on matrix algebras, a classical theorem by Choi [4] is important in determining complete positivity. Before that, we recall the notion of the Choi matrix for a linear map.

Definition 2.1. Let $B(K)$ and $B(H)$ denote the space of bounded linear operators on finite dimensional Hilbert spaces K and H , respectively. Let $E_{ij}, i, j = 1, \dots, m$, be the canonical matrix units for $B(K)$ and $(\dim(K), \dim(H)) = (m, n)$. Given a linear map $\phi \in B(B(K), B(H))$, the Choi matrix C_ϕ for ϕ is:

$$C_\phi \triangleq \sum_{i,j=1}^m E_{ij} \otimes \phi(E_{ij}) = [\phi(E_{ij})]_{i,j=1}^m \in M_m(M_n(\mathbb{C})).$$

Remark 2.2. Obviously the map $\phi \mapsto C_\phi$ is a bijection between linear maps in $B(M_m(\mathbb{C}), M_n(\mathbb{C}))$ and matrices in $M_m(M_n(\mathbb{C}))$ which preserves linearity.

Theorem 2.3 (Choi, 1975). *A positive map $\phi \in B(B(K), B(H))$ is completely positive if and only if the corresponding Choi matrix is positive.*

The peel-off theorem first appeared in [9] (see also Størmer's book [12, pages 38-39]). Combined with Zorn's Lemma Størmer obtained a decomposition for positive maps in [13]. Here we present a slightly stronger version (Theorem 2.7) of the peel-off result by block-matrix approach, which was shown by M.-D. Choi for the case of 2-positive maps [5]. Let us consider k -positive maps for the moment. A similar theorem holds for k -copositive maps.

Definition 2.4 (Trivial Lifting). Given a linear map $\chi \in B(M_s(\mathbb{C}), M_n(\mathbb{C}))$, fix the canonical matrix unit basis $E_{ij}, i, j = 1, \dots, s$, in $M_s(\mathbb{C})$, under which the Choi matrix is $C_\chi = [\chi(E_{ij})]_{i,j=1}^s \in M_s(M_n(\mathbb{C}))$. Given $I = \{n_1, \dots, n_p\} \subset \{1, \dots, s+p\}$, where $n_1 < \dots < n_p$, extend the matrix C_χ to a $(s+p) \times (s+p)$ block matrix $C_I^{lift} \in M_{s+p}(M_n(\mathbb{C}))$ by adding one row and one column of $n \times n$ zero matrices at the n_k^{th} level for each $k = 1, \dots, p$ as follows:

$$C_I^{lift} \triangleq n_k^{th} \begin{pmatrix} 1^{st} & \cdots & n_k^{th} & \cdots & (s+p)^{th} \\ \chi(E_{11}) & \cdots & 0 & \cdots & \chi(E_{1,s}) \\ \vdots & \ddots & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 \\ \vdots & \ddots & 0 & \ddots & \vdots \\ \chi(E_{s,1}) & \cdots & 0 & \cdots & \chi(E_{s,s}) \end{pmatrix}.$$

Denote by $\tilde{\chi}_I$ the map in $B(M_{s+p}(\mathbb{C}), M_n(\mathbb{C}))$ associated with the Choi matrix $C_{\tilde{\chi}_I} = [\tilde{\chi}_p(E_{ij})]_{i,j=1}^{s+p} = C_I^{lift}$. Then the map $\tilde{\chi}_I$ is called a I-trivial lifting of the original map χ . If $I = \{q\}$ is a singleton, simply denote by $\tilde{\chi}_q$ the q -trivial lifting of χ .

Lemma 2.5. *The map χ is k -positive or k -copositive if and only if the trivial lifting $\tilde{\chi}_p$ is k -positive or k -copositive, respectively.*

Proof. Let $\eta = (w^1, \dots, w^k)^T$ be an arbitrary vector in $\mathbb{C}^k \otimes \mathbb{C}^m$ where $w^s \in \mathbb{C}^m$, $s = 1, \dots, k$. Let $\hat{w}^s \in \mathbb{C}^{m-1}$ be defined as $(w_1^s, \dots, w_{p-1}^s, w_{p+1}^s, \dots, w_m^s)^T$ for $s = 1, \dots, k$, and $\hat{\eta} = (\hat{w}^1, \dots, \hat{w}^k) \in \mathbb{C}^k \otimes \mathbb{C}^{m-1}$. By definition of p -trivial lifting,

$$(id_k \otimes \tilde{\chi}_p)(\eta \eta^*) = [\tilde{\chi}_p(w^s (w^t)^*)]_{s,t=1}^k = [\chi(\hat{w}^s (\hat{w}^t)^*)]_{s,t=1}^k = (id_k \otimes \chi)(\hat{\eta} \hat{\eta}^*).$$

This matrix equality in $M_k(M_n(\mathbb{C}))$ shows that the pair of maps $(\chi, \tilde{\chi}_p)$ are k -positive simultaneously. For k -copositivity, we also have:

$$(\tau_k \otimes \tilde{\chi}_p)(\eta \eta^*) = [\tilde{\chi}_p(w^t (w^s)^*)]_{s,t=1}^k = [\chi(\hat{w}^t (\hat{w}^s)^*)]_{s,t=1}^k = (\tau_k \otimes \chi)(\hat{\eta} \hat{\eta}^*),$$

which completes the proof. \square

Remark 2.6. By repeatedly using Lemma 2.5, a map χ is k -positive or k -copositive, if and only if the trivial lifting $\tilde{\chi}_I$ is k -positive or k -copositive, respectively.

Theorem 2.7. (Choi Decomposition) *Let ϕ be a non-zero k -positive ($2 \leq k < \min\{m, n\}$) map in $B(M_m(\mathbb{C}), M_n(\mathbb{C}))$. Then there exists a decomposition $\phi = \psi + \gamma$, where ψ is a non-zero completely positive map and γ is a p -trivial lifting of a $(k-1)$ -positive map in $B(M_{m-1}(\mathbb{C}), M_n(\mathbb{C}))$, for some $p \in \{1, \dots, m\}$.*

Before proving Theorem 2.7, recall a classical result (see [2, Exercise 1.3.5]):

Lemma 2.8. *Suppose a hermitian matrix M is partitioned as*

$$M = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix},$$

where A and C are square matrices. Then the following conditions are equivalent:

- (1) $M \geq 0$.
- (2) $A \geq 0$, $M/A = C - B^* A^\dagger B \geq 0$, $\text{range}(B) \subseteq \text{range}(A)$.
- (3) $C \geq 0$, $M/C = A - B C^\dagger B^* \geq 0$, $\text{range}(B^*) \subseteq \text{range}(C)$.

Here A^\dagger and C^\dagger refer to the Moore-Penrose pseudo inverses of A and C , respectively.

Remark 2.9. Recall some properties of the Moore-Penrose pseudo inverse A^\dagger of a matrix A (see [1, pages 29-30]):

- P1. $AA^\dagger A = A$, $A^\dagger AA^\dagger = A^\dagger$.
- P2. $(AA^\dagger)^* = AA^\dagger$, $(A^\dagger A)^* = A^\dagger A$.

- P3. AA^\dagger is the orthogonal projector onto the range of A , $A^\dagger A$ is the orthogonal projector onto the range of A^* .
- P4. If A is invertible, then $A^\dagger = A^{-1}$.
- P5. If $A \geq 0$, then $A^\dagger \geq 0$.

Proof. (of Theorem 2.7) Since the k -positive map $\phi \neq 0$, with respect to the canonical matrix units E_{ij} , $i, j = 1, \dots, m$, in $M_m(\mathbb{C})$, there exists an index $k \in \{1, 2, \dots, m\}$ such that $\phi(E_{kk}) \neq 0$. Otherwise if $\phi(E_{kk}) = 0$ for every $k = 1, \dots, m$, then $\phi(I_m) = 0$. Meanwhile for every $A \in M_m(\mathbb{C})^+$, $\|A\|I_m - A \geq 0$ yields that $0 = \|A\|\phi(I_m) \geq \phi(A)$, implying $\phi = 0$, which contradicts $\phi \neq 0$. Without loss of generality, we assume that $\phi(E_{mm}) \neq 0$. Decompose the Choi matrix C_ϕ for ϕ , with $A_{ij} = \phi(E_{ij})$, $i, j = 1, \dots, m$, as follows:

$$\begin{aligned}
C_\phi &= \begin{pmatrix} A_{11} & \cdots & A_{1j} & \cdots & A_{1m} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{i1} & \cdots & A_{ij} & \cdots & A_{im} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mj} & \cdots & A_{mm} \end{pmatrix} \\
&= \begin{pmatrix} A_{1m}A_{mm}^\dagger A_{m1} & \cdots & A_{1m}A_{mm}^\dagger A_{mj} & \cdots & A_{1m}A_{mm}^\dagger A_{mm} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{im}A_{mm}^\dagger A_{m1} & \cdots & A_{im}A_{mm}^\dagger A_{mj} & \cdots & A_{im}A_{mm}^\dagger A_{mm} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{mm}A_{mm}^\dagger A_{m1} & \cdots & A_{mm}A_{mm}^\dagger A_{mj} & \cdots & A_{mm}A_{mm}^\dagger A_{mm} \end{pmatrix} \\
&+ \begin{pmatrix} A_{11} - A_{1m}A_{mm}^\dagger A_{m1} & \cdots & A_{1j} - A_{1m}A_{mm}^\dagger A_{mj} & \cdots & A_{1m} - A_{1m}A_{mm}^\dagger A_{mm} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{i1} - A_{im}A_{mm}^\dagger A_{m1} & \cdots & A_{ij} - A_{im}A_{mm}^\dagger A_{mj} & \cdots & A_{im} - A_{im}A_{mm}^\dagger A_{mm} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{m1} - A_{mm}A_{mm}^\dagger A_{m1} & \cdots & A_{mj} - A_{mm}A_{mm}^\dagger A_{mj} & \cdots & A_{mm} - A_{mm}A_{mm}^\dagger A_{mm} \end{pmatrix} \\
&\triangleq U + R = C_\psi + C_\gamma.
\end{aligned}$$

For $i, j = 1, \dots, m$, the (i, j) -entry of the matrix U is given by $A_{im}A_{mm}^\dagger A_{mj}$, and the (i, j) -entry of the matrix R is given by $R_{ij} = A_{ij} - A_{im}A_{mm}^\dagger A_{mj}$. Note that

$$U = \begin{pmatrix} A_{1m} \\ \vdots \\ A_{im} \\ \vdots \\ A_{mm} \end{pmatrix} A_{mm}^\dagger \begin{pmatrix} A_{m1} & \cdots & A_{mj} & \cdots & A_{mm} \end{pmatrix} \geq 0$$

and $U \neq 0$, since its (m, m) -entry is $A_{mm}A_{mm}^\dagger A_{mm} = A_{mm} = \phi(E_{mm}) \neq 0$. Then the map $\psi \neq 0$ corresponding to the matrix U is completely positive. By employing k -positivity of ϕ , for arbitrary column vectors $w^1, w^2, \dots, w^{k-1} \in \mathbb{C}^m$, taking $\xi = (w^1, \dots, w^{k-1}, e_m)^T$ where $e_m = (0, \dots, 0, 1)^T \in \mathbb{C}^m$,

$$\begin{aligned} \xi \xi^* &= \begin{pmatrix} w^1(w^1)^* & \cdots & w^1(w^j)^* & \cdots & w^1 e_m^* \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ w^i(w^1)^* & \cdots & w^i(w^j)^* & \cdots & w^i e_m^* \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ e_m(w^1)^* & \cdots & e_m(w^j)^* & \cdots & e_m e_m^* \end{pmatrix} \geq 0 \\ \implies (id_k \otimes \phi)(\xi \xi^*) &= \begin{pmatrix} \phi(w^1(w^1)^*) & \cdots & \phi(w^1(w^j)^*) & \cdots & \phi(w^1 e_m^*) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \phi(w^i(w^1)^*) & \cdots & \phi(w^i(w^j)^*) & \cdots & \phi(w^i e_m^*) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \phi(e_m(w^1)^*) & \cdots & \phi(e_m(w^j)^*) & \cdots & \phi(e_m e_m^*) \end{pmatrix} \geq 0. \end{aligned}$$

By Lemma 2.8 (3), the condition $(id_k \otimes \phi)(\xi \xi^*) \geq 0$ expands to:

$$\begin{aligned} &\begin{pmatrix} \phi(w^1(w^1)^*) & \cdots & \phi(w^1(w^{k-1})^*) \\ \vdots & \ddots & \vdots \\ \phi(w^{k-1}(w^1)^*) & \cdots & \phi(w^{k-1}(w^{k-1})^*) \end{pmatrix} \geq \begin{pmatrix} \phi(w^1 e_m^*) \\ \vdots \\ \phi(w^{k-1} e_m^*) \end{pmatrix} \phi(e_m e_m^*)^\dagger \begin{pmatrix} \phi(e_m(w^1)^*) & \cdots & \phi(e_m(w^{k-1})^*) \end{pmatrix} \\ \iff &\begin{pmatrix} \phi(w^1(w^1)^*) - \phi(w^1 e_m^*) \phi(e_m e_m^*)^\dagger \phi(e_m(w^1)^*) & \cdots & \phi(w^1(w^{k-1})^*) - \phi(w^1 e_m^*) \phi(e_m e_m^*)^\dagger \phi(e_m(w^{k-1})^*) \\ \vdots & \ddots & \vdots \\ \phi(w^{k-1}(w^1)^*) - \phi(w^{k-1} e_m^*) \phi(e_m e_m^*)^\dagger \phi(e_m(w^1)^*) & \cdots & \phi(w^{k-1}(w^{k-1})^*) - \phi(w^{k-1} e_m^*) \phi(e_m e_m^*)^\dagger \phi(e_m(w^{k-1})^*) \end{pmatrix} \geq 0. \end{aligned}$$

For the (s, t) entry in the above matrix, by linearity,

$$\begin{aligned} &\phi(w^s e_m^*) \phi(e_m e_m^*)^\dagger \phi(e_m(w^t)^*) \\ &= \phi\left(\sum_{i=1}^m w_i^s e_i e_m^*\right) \phi(e_m e_m^*)^\dagger \phi\left(\sum_{j=1}^m \overline{w_j^t} e_m e_j^*\right) \\ &= \left(\sum_{i=1}^m w_i^s \phi(E_{im})\right) \phi(E_{mm})^\dagger \left(\sum_{j=1}^m \overline{w_j^t} \phi(E_{mj})\right) \\ &= \sum_{i=1}^m \sum_{j=1}^m w_i^s \overline{w_j^t} \left(\phi(E_{im}) \phi(E_{mm})^\dagger \phi(E_{mj})\right) \\ &= \sum_{i=1}^m \sum_{j=1}^m w_i^s \overline{w_j^t} (A_{im} A_{mm}^\dagger A_{mj}) \\ &= \sum_{i=1}^m \sum_{j=1}^m w_i^s \overline{w_j^t} U_{ij} \\ &= \sum_{i=1}^m \sum_{j=1}^m w_i^s \overline{w_j^t} \psi(e_i e_j^*) \\ &= \psi(w^s (w^t)^*). \end{aligned}$$

Since $\gamma = \phi - \psi$, one has

$$\begin{pmatrix} \gamma(w^1(w^1)^*) & \cdots & \gamma(w^1(w^{k-1})^*) \\ \vdots & \ddots & \vdots \\ \gamma(w^{k-1}(w^1)^*) & \cdots & \gamma(w^{k-1}(w^{k-1})^*) \end{pmatrix} \geq 0, \forall w^1, \dots, w^{k-1} \in \mathbb{C}^m,$$

proving that γ is $(k-1)$ -positive. Moreover, all the entries of the m^{th} row and m^{th} column of the matrix R are zero matrices. To show this, recall that ϕ is 2-positive ($k \geq 2$), hence any sub-block $\begin{pmatrix} \phi(E_{mm}) & \phi(E_{mj}) \\ \phi(E_{jm}) & \phi(E_{jj}) \end{pmatrix} \geq 0$, for all $j = 1, \dots, m-1$. By Lemma 2.8, one obtains that $\text{range}(\phi(E_{mj})) \subseteq \text{range}(\phi(E_{mm}))$, for all $j = 1, \dots, m$. By property P3 in Remark 2.9, $A_{mm}A_{mm}^\dagger$ is the orthogonal projector onto the range of A_{mm} , so $R_{mj} = A_{mj} - A_{mm}A_{mm}^\dagger A_{mj} = 0$, for all $j = 1, \dots, m$. Denote the matrix $R = C_\gamma$ by:

$$R = \begin{pmatrix} & 0 \\ K & \vdots \\ 0 & \dots & 0 \end{pmatrix} = \begin{pmatrix} & 0 \\ C_\kappa & \vdots \\ 0 & \dots & 0 \end{pmatrix}.$$

Here, the map $\kappa \in B(M_{m-1}(\mathbb{C}), M_n(\mathbb{C}))$ is defined by the Choi matrix $K \in M_{(m-1)n}(\mathbb{C})$ through $\kappa(E_{st}) = K_{st}$, $s, t = 1, \dots, m-1$. It is obvious that $\gamma \in B(M_m(\mathbb{C}), M_n(\mathbb{C}))$ is the m -trivial lifting of $\kappa \in B(M_{m-1}(\mathbb{C}), M_n(\mathbb{C}))$. By Lemma 2.5, the map $\kappa \in B(M_{m-1}(\mathbb{C}), M_n(\mathbb{C}))$ is $(k-1)$ -positive. \square

A similar result holds for k -copositive maps.

Corollary 2.10. *Let ϕ be a non-zero k -copositive ($2 \leq k < \min\{m, n\}$) map in $B(M_m(\mathbb{C}), M_n(\mathbb{C}))$. Then there exists a decomposition $\phi = \psi + \gamma$, where ψ is a non-zero completely copositive map and γ is a p -trivial lifting of a $(k-1)$ -copositive map in $B(M_{m-1}(\mathbb{C}), M_n(\mathbb{C}))$, for some $p \in \{1, \dots, m\}$.*

Proof. If ϕ is k -copositive, using the same arguments in proof of Theorem 2.7 for the matrix $\sum_{i,j=1}^m e_{ji} \otimes \phi(e_{ij})$, one obtains a decomposition $\sum_{i,j=1}^m e_{ji} \otimes \phi(e_{ij}) = \sum_{i,j=1}^m e_{ji} \otimes \psi(e_{ij}) + \sum_{i,j=1}^m e_{ji} \otimes \gamma(e_{ij})$, where ψ is a non-zero completely copositive map and γ is a $(k-1)$ -copositive map which is a trivial lifting of a map in $B(M_{m-1}(\mathbb{C}), M_n(\mathbb{C}))$. \square

Theorem 2.11. *Let $2 \leq k < \min\{m, n\}$. Any non-zero k -positive (respectively k -copositive) map in $B(M_m(\mathbb{C}), M_n(\mathbb{C}))$ is the sum of at most $(k-1)$ many non-zero completely positive (respectively completely copositive) maps and a positive map which is the trivial lifting of a positive map in $B(M_{m-k+1}(\mathbb{C}), M_n(\mathbb{C}))$.*

Proof. For a k -positive linear map ϕ , repeatedly using Theorem 2.7 (respectively Corollary 2.10) until the remainder is a positive map. \square

The Choi decomposition may no longer be valid for a general positive map ϕ even when ϕ is in $B(M_2(\mathbb{C}), M_2(\mathbb{C}))$. And it may not necessarily give us an algorithm to decompose a positive map in $B(M_2(\mathbb{C}), M_2(\mathbb{C}))$ as the sum of a completely positive map and a completely copositive map. Let us illustrate this by a simple example in $B(M_2(\mathbb{C}), M_2(\mathbb{C}))$.

Example 2.12. Let ε be a real number and ω in $B(M_2(\mathbb{C}), M_2(\mathbb{C}))$ be defined through its Choi matrix:

$$C_\omega = \begin{pmatrix} 1 & 0 & 0 & \varepsilon \\ 0 & 0 & \varepsilon & 0 \\ 0 & \varepsilon & 0 & 0 \\ \varepsilon & 0 & 0 & 1 \end{pmatrix},$$

Hence the map ω is given by

$$\omega \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & \varepsilon(b+c) \\ \varepsilon(b+c) & d \end{bmatrix}, \quad a, b, c, d \in \mathbb{C}.$$

For ω to be positive, it suffices to show for any vector $y = (y_1, y_2)^T \in \mathbb{C}^2$, the matrix

$$\begin{aligned} & |y_1|^2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + y_1 \overline{y_2} \begin{pmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{pmatrix} + y_2 \overline{y_1} \begin{pmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{pmatrix} + |y_2|^2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} |y_1|^2 & 2\varepsilon \operatorname{Re}(y_1 \overline{y_2}) \\ 2\varepsilon \operatorname{Re}(y_1 \overline{y_2}) & |y_2|^2 \end{pmatrix} \end{aligned}$$

is positive. This is equivalent to the condition that $-\frac{1}{2} \leq \varepsilon \leq \frac{1}{2}$. For Choi decomposition, using $A_{11}^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, we have

$$C_\omega = \begin{pmatrix} 1 & 0 & 0 & \varepsilon \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \varepsilon & 0 & 0 & \varepsilon^2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon & 0 \\ 0 & \varepsilon & 0 & 0 \\ 0 & 0 & 0 & 1 - \varepsilon^2 \end{pmatrix},$$

and using $A_{22}^\dagger = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, we have

$$C_\omega = \begin{pmatrix} \varepsilon^2 & 0 & 0 & \varepsilon \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \varepsilon & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 - \varepsilon^2 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon & 0 \\ 0 & \varepsilon & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

In each of the two equations above, the last matrix corresponds to a linear map which is not positive. Meanwhile to decompose the map ω as the sum of a completely positive map and a completely copositive map, one splits the original matrix as follows:

$$C_\omega = \begin{pmatrix} 1/2 & 0 & 0 & \varepsilon \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \varepsilon & 0 & 0 & 1/2 \end{pmatrix} + \begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon & 0 \\ 0 & \varepsilon & 0 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix}.$$

Obviously under this splitting, the second and the third matrix in the above equation correspond to a completely positive map ψ_1 and a completely copositive map ψ_2 , respectively,

where $\psi_1 \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \frac{a}{2} & \varepsilon b \\ \varepsilon c & \frac{d}{2} \end{bmatrix}$ and $\psi_2 \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \frac{a}{2} & \varepsilon c \\ \varepsilon b & \frac{d}{2} \end{bmatrix}$.

3. A REDUCED SITUATION IN $B(M_3(\mathbb{C}), M_3(\mathbb{C}))$

In low dimensional cases such as $B(M_2(\mathbb{C}), M_2(\mathbb{C}))$ and $B(M_2(\mathbb{C}), M_3(\mathbb{C}))$, Woronowicz and Størmer respectively showed that every positive map is decomposable (see [15]). In this section, we will show that in $B(M_3(\mathbb{C}), M_3(\mathbb{C}))$, although positive maps may not be decomposable, 2-positive maps are always decomposable. Let us start with a useful lemma. For any $p \in \{1, \dots, m\}$, we assume that $\tilde{\chi}_p \in B(M_m(\mathbb{C}), M_n(\mathbb{C}))$ is the p -trivial lifting of a positive map $\chi \in B(M_{m-1}(\mathbb{C}), M_n(\mathbb{C}))$.

Lemma 3.1. *If χ is decomposable in $B(M_{m-1}(\mathbb{C}), M_n(\mathbb{C}))$, then its trivial lifting $\tilde{\chi}_p$ is also decomposable in $B(M_m(\mathbb{C}), M_n(\mathbb{C}))$.*

Proof. Given a decomposable map $\chi \in B(M_{m-1}(\mathbb{C}), M_n(\mathbb{C}))$, then $\chi = \chi^1 + \chi^2$, where χ^1 is completely positive and χ^2 is completely copositive. By Lemma 2.5, one obtains a completely positive map $\widetilde{\chi^1}_p$ and a completely copositive map $\widetilde{\chi^2}_p$ through p -trivial lifting

of χ^1 and χ^2 , respectively. By linearity of the trivial lifting, $\tilde{\chi}_p = \widetilde{(\chi^1 + \chi^2)}_p = \widetilde{\chi^1}_p + \widetilde{\chi^2}_p$ is decomposable in $B(M_m(\mathbb{C}), M_n(\mathbb{C}))$. \square

The next result gives an affirmative answer to Conjecture 1.1.

Theorem 3.2. *Every 2-positive or 2-copositive map ϕ in $B(M_3(\mathbb{C}), M_3(\mathbb{C}))$ is decomposable.*

Proof. Without loss of generality, we assume the 2-positive(respectively 2-copositive) map ϕ is not zero. In this concrete case of $B(M_3(\mathbb{C}), M_3(\mathbb{C}))$, the peel-off process yields that:

$$\phi = \psi + \tilde{\kappa}_p \text{ for some } p \in \{1, \dots, m\}$$

where ψ is completely positive (respectively completely copositive) and $\tilde{\kappa}_p$ is a p-trivial lifting of a positive map $\kappa \in B(M_2(\mathbb{C}), M_3(\mathbb{C}))$. Since every positive map in $B(M_2(\mathbb{C}), M_3(\mathbb{C}))$ is decomposable in $B(M_2(\mathbb{C}), M_3(\mathbb{C}))$ (see [15]), by Lemma 3.1, the lifted map $\tilde{\kappa}_p$ is decomposable in $B(M_3(\mathbb{C}), M_3(\mathbb{C}))$. Hence, $\phi = \psi + \tilde{\kappa}_p$ is also decomposable. \square

Definition 3.3. A positive linear map in $B(M_m(\mathbb{C}), M_n(\mathbb{C}))$ is called atomic if it is not the sum of a 2-positive map and a 2-copositive map.

Remark 3.4. From the definition, an atomic map in $B(M_m(\mathbb{C}), M_n(\mathbb{C}))$ is indecomposable. The converse is true when $m = n = 3$.

Corollary 3.5. *Every indecomposable map in $B(M_3(\mathbb{C}), M_3(\mathbb{C}))$ is atomic.*

Corollary 3.6. *Under the dual cone correspondence (see [8]), one can completely determine the set inclusion relations in $B(M_3(\mathbb{C}), M_3(\mathbb{C}))$ as follows:*

$$\begin{array}{ccccccc} \mathbb{V}_1 & \subsetneq & \mathbb{T} & \subsetneq & \mathbb{V}_2 & \subsetneq & \mathbb{V}_3 = (M_3(\mathbb{C}) \otimes M_3(\mathbb{C}))^+ \\ \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\ \mathbb{P}_1 & \supsetneq & \mathbb{D} & \supsetneq & \mathbb{P}_2 & \supsetneq & \mathbb{P}_3 \cong (M_3(\mathbb{C}) \otimes M_3(\mathbb{C}))^+ \end{array}$$

Here we denote by \mathbb{V}_k the set of all quantum states of Schmidt number k , \mathbb{P}_k the set of all k -positive maps, \mathbb{D} the cone of all decomposable maps and \mathbb{T} the cone of all positive partial transpose states.

Remark 3.7. The inclusion $\mathbb{T} \subset \mathbb{V}_2$ gives an affirmative answer to Conjecture 1.2, stating that all positive partial transpose entangled states in 3×3 system have Schmidt number 2.

Example 3.8. We will illustrate Choi decomposition using the 2-positive generalized Choi maps $\Phi[a, b, c]$ defined in [3] by

$$\Phi[a, b, c](X) = \begin{pmatrix} ax_{11} + bx_{22} + cx_{33} & -x_{12} & -x_{13} \\ -x_{21} & cx_{11} + ax_{22} + bx_{33} & -x_{23} \\ -x_{31} & -x_{32} & bx_{11} + cx_{22} + ax_{33} \end{pmatrix}$$

for $X = [x_{ij}] \in M_3(\mathbb{C}^3)$, where a, b, c, d are nonnegative numbers. Note that $\Phi[a, b, c]$ is 2-positive if and only if $a \geq 2$ or $[1 \leq a < 2] \wedge [bc \geq (2-a)(b+c)]$. Let us consider the non-trivial case when the map $\Phi[a, b, c]$ is 2-positive but not completely positive. Hence $a \in [1, 2)$ and $bc \geq (2-a)(b+c)$, which imply that

$$(*) \quad bc \geq 4(2-a)^2 \geq (a-1-\frac{2}{a})^2.$$

The Choi matrix of the map $\Phi[a, b, c]$ is

$$C_{\Phi[a,b,c]} = \begin{pmatrix} a & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & a & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & a \end{pmatrix}.$$

Using $A_{11}^\dagger \triangleq \Phi[a,b,c](E_{11})^\dagger$, the Choi decomposition for the 2-positive map $\Phi[a,b,c]$ is

$$C_{\Phi[a,b,c]} = \begin{pmatrix} a & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & \frac{1}{a} & 0 & 0 & 0 & \frac{1}{a} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & \frac{1}{a} & 0 & 0 & 0 & \frac{1}{a} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a - \frac{1}{a} & 0 & 0 & 0 & -1 - \frac{1}{a} \\ 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & -1 - \frac{1}{a} & 0 & 0 & 0 & a - \frac{1}{a} \end{pmatrix}.$$

By 1-trivial lifting, the last matrix can be regarded as the Choi matrix for a positive linear map Υ in $B(M_2(\mathbb{C}), M_3(\mathbb{C}))$. While every positive map in $B(M_2(\mathbb{C}), M_3(\mathbb{C}))$ is decomposable, one obtains a decomposition of Υ as follows:

$$\begin{aligned} C_\Upsilon &= \begin{pmatrix} b & 0 & 0 & 0 & 0 & 0 \\ 0 & a - \frac{1}{a} & 0 & 0 & 0 & -1 - \frac{1}{a} \\ 0 & 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & b & 0 \\ 0 & -1 - \frac{1}{a} & 0 & 0 & 0 & a - \frac{1}{a} \end{pmatrix} \\ &= \begin{pmatrix} b & 0 & 0 & 0 & 0 & 0 \\ 0 & a - \frac{1}{a} & 0 & 0 & 0 & \frac{1}{a} - a \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{a} - a & 0 & 0 & 0 & a - \frac{1}{a} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a - 1 - \frac{2}{a} \\ 0 & 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b & 0 \\ 0 & a - 1 - \frac{2}{a} & 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Combining with the Choi decomposition of $C_{\Phi[a,b,c]}$, we have

$$C_{\Phi[a,b,c]} = \begin{pmatrix} a & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & a & 0 & 0 & 0 & \frac{2}{a} - a \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & \frac{2}{a} - a & 0 & 0 & 0 & a \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a - 1 - \frac{2}{a} \\ 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & a - 1 - \frac{2}{a} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence $\Phi[a, b, c] = \Phi_1 + \Phi_2$, where

$$\Phi_1 \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} ax_{11} + bx_{22} + cx_{33} & -x_{12} & -x_{13} \\ -x_{21} & cx_{11} + ax_{22} & (\frac{2}{a} - a)x_{23} \\ -x_{31} & (\frac{2}{a} - a)x_{32} & bx_{11} + ax_{33} \end{bmatrix},$$

$$\Phi_2 \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & bx_{33} & (a - 1 - \frac{2}{a})x_{23} \\ 0 & (a - 1 - \frac{2}{a})x_{32} & cx_{22} \end{bmatrix}.$$

Since $\Phi[a, b, c]$ is 2-positive but not completely positive, by condition (*) the matrices C_{Φ_1} and partial transpose of C_{Φ_2} are positive, implying that Φ_1 is completely positive and Φ_2 is completely copositive, respectively. Note that our method of writing the 2-positive map $\Phi[a, b, c]$ as a sum of a completely positive map and a completely copositive map differs from another method mentioned in [3, proof of Theorem 3.4]:

$$\Phi[a, b, c] = (1 - \sqrt{bc})\Phi\left[\frac{a - \sqrt{bc}}{1 - \sqrt{bc}}, 0, 0\right] + \sqrt{bc}\Phi\left[1, \sqrt{\frac{b}{c}}, \sqrt{\frac{c}{b}}\right].$$

So the decomposition of the 2-positive map $\Phi[a, b, c]$ into a sum of a completely positive map and a completely copositive map is not unique.

In view of the previous results, it is natural to pose the following

Question 3.9. *Does there exist a 2-positive but indecomposable map in $B(M_3(\mathbb{C}), M_4(\mathbb{C}))$?*

Most likely this question has an affirmative answer.

4. ACKNOWLEDGEMENT

The authors are indebted to Professor M.-D. Choi for his sharing of the decomposition of the Choi matrix for a 2-positive map [5]. Subsequently, Professor M.-D. Choi and the authors independently proved Conjecture 1.1. This work is partially supported by Singapore Ministry of Education Academic Research Fund Tier 1 Grant (No. R-146-003-193-112).

REFERENCES

- [1] A. Ben-Israel and T. N. E. Greville, *Generalized Inverses: Theory and Applications*, CMS Books in Mathematics, 2003.
- [2] R. Bhatia, *Positive Definite Matrices*, Princeton Series in Applied Mathematics, 2007.
- [3] S. J. Cho, S.-H. Kye and S. G. Lee, *Generalized Choi maps in three-dimensional matrix algebra*, Linear Alg. Appl. **171** (1992), 213-224.
- [4] M.-D. Choi, *Completely positive linear maps on complex matrices*, Linear Alg. Appl. **12** (1975), 285-290.
- [5] M.-D. Choi, Seminar talk, Department of Mathematics, National University of Singapore, 30 July 2015.
- [6] M.-H. Eom and S.-H. Kye, *Duality for positive linear maps in matrix algebras*, Math. Scan. **86** (2000), 130-142.
- [7] T. Itoh, *Positive maps and cones in C^* -algebras*, Math. Japonica, **31** (1986), 607-616.
- [8] S.-H. Kye, *Facial structures for various notions of positivity and applications to the theory of entanglement*, Rev. Math. Phys. **25** (2013), 1330002.
- [9] M. Marciniak, *On extremal positive maps acting between type I factors*, Noncommutative Harmonic Analysis with Applications to probability II, Banach Center Publications, **89** (2010), 201-221.
- [10] A. Sanpera, D. Bruß and M. Lewenstein, *Schmidt number witnesses and bound entanglement*, Phys. Rev. A **63** (2001), 050301.
- [11] E. Størmer, *Extension of positive maps into $B(\mathcal{H})$* , J. of Funct. Anal. **66** (1986), 235-254.
- [12] E. Størmer, *Positive Linear Maps of Operator Algebras*, Springer Monographs in Mathematics, 2010.
- [13] E. Størmer, *A decomposition theorem for positive maps, and the projection onto a spin factor*, arXiv:1308.3332 preprint, to appear in Math. Scan..

- [14] B. M. Terhal and P. Horodecki, *A Schmidt number for density matrices*, Phys Rev. A **61** (2000), 040301.
- [15] S. L. Woronowicz, *Positive maps of low dimensional matrix algebras*, Rep. Math. Phys. **10** (1976), 165-183.

(Yu Yang) DEPARTMENT OF MATHEMATICS, NATIONAL UNIVERSITY OF SINGAPORE, 10 LOWER KENT RIDGE ROAD, SINGAPORE 119076, REPUBLIC OF SINGAPORE
E-mail address: a0086285@nus.edu.sg

(Denny H. Leung) DEPARTMENT OF MATHEMATICS, NATIONAL UNIVERSITY OF SINGAPORE, 10 LOWER KENT RIDGE ROAD, SINGAPORE 119076, REPUBLIC OF SINGAPORE
E-mail address: mat1hh@nus.edu.sg

(Wai-Shing Tang) DEPARTMENT OF MATHEMATICS, NATIONAL UNIVERSITY OF SINGAPORE, 10 LOWER KENT RIDGE ROAD, SINGAPORE 119076, REPUBLIC OF SINGAPORE
E-mail address: mattws@nus.edu.sg